Chapterwise weightage of Marks : 08

4 Addition Formulae and Factor Formulae

We will study following topics of JEE syllabus in this chapter.

Addition and factor formulae : Exercise : 4.1
Exercise : 4.2
Exercise : 4.3, Exercise : 4.0

Remember

If $f(x) = ax$, $x \in R$ is a linear function then $f(x-y) = a(x-y) = ax - ay = f(x) - f(y)$

$\Rightarrow f(x-y) = f(x) - f(y)$

Theorem 1 : For $\alpha, \beta \in R$,

(i) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ **(ii)** $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Corollary : 1 (i) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ **(ii)** $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

Theorem : 2 (i) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ **(ii)** $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Some formulae :

Y $\frac{\pi}{2}$

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

$\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$

$\tan(\pi-\theta) = -\tan\theta$

$\csc\left(\frac{\pi}{2}+\theta\right) = \sin\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

$\sin(2\pi+\theta) = \sin\theta$

$\cos(2\pi+\theta) = \cos\theta$

$\tan(2\pi+\theta) = \tan\theta$

X' π 2π X

$\sin(\pi+\theta) = -\sin\theta$

$\cos(\pi+\theta) = -\cos\theta$

$\tan(\pi+\theta) = \tan\theta$

$\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$

$\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta$

$\cot\left(\frac{3\pi}{2}-\theta\right) = \tan\theta$

$\sin(2\pi-\theta) = -\sin\theta$

$\cos(2\pi-\theta) = \cos\theta$

$\tan(2\pi-\theta) = -\tan\theta$

$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$

$\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$

$\sec\left(\frac{3\pi}{2}+\theta\right) = \csc\theta$

Y' $\frac{3\pi}{2}$

❖ **Some important results :**

$$\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}, \quad \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sin\frac{5\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}, \quad \cos\frac{5\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(i) $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$

(ii) $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$

❖ $f(\alpha) = a\cos\alpha + b\sin\alpha,\ a, b \in R,\ a^2 + b^2 \neq 0$

(i) If $a = 0,\ b \neq 0$ then range of $f(\alpha) = [-|b|, |b|]$.

(ii) If $a \neq 0,\ b = 0$ then range of $f(\alpha) = [-|a|, |a|]$.

(iii) If $a \neq 0,\ b \neq 0$ then range of $f(\alpha) = [-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$.

❖ **Addition formulae of tan and cot functions :**

(i) $\alpha, \beta, \alpha+\beta \in R - \left\{(2k-1)\frac{\pi}{2} / k \in z\right\}$ then, $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ and

$\alpha, \beta, \alpha-\beta \in R \left\{(2k-1)\frac{\pi}{2} / k \in z\right\}$ then, $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

(ii) $\alpha, \beta, \alpha+\beta \in R - \{k\pi / k \in Z\}$ then, $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$ and

$\alpha, \beta, \alpha-\beta \in R - \{k\pi / k \in Z\}$ then, $\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

❖ $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ and $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$

$\cot\frac{\pi}{12} = 2 + \sqrt{3}$ and $\cot\frac{5\pi}{12} = 2 - \sqrt{3}$

❖ **Expression of product in the form of a sum or a difference and Expression of the sum or difference as a product.**

(i) $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

(ii) $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\cos\beta$

(iii) $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

(iv) $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$

❖ **(i)** $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iii) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Exercise 4.1 [Textbook Page No. 70]

1. Evaluate :

(1) cos135°

$= \cos(180° - 45°)$

$= -\cos 45°$ (second quadrant)

$= -\frac{1}{\sqrt{2}}$

(2) $\tan\left(-\frac{23\pi}{6}\right)$

$= -\tan\left(\frac{23\pi}{6}\right)$ ($\because$ tan is an odd function)

$= -\left\{\tan\left(\frac{24\pi - \pi}{6}\right)\right\}$

$= -\left\{\tan\left(4\pi - \frac{\pi}{6}\right)\right\}$

$= -\left\{-\tan\frac{\pi}{6}\right\}$

($\because$ $4\pi - \frac{\pi}{6}$ is in the fourth quadrant)

$= -\left\{-\frac{1}{\sqrt{3}}\right\} = \frac{1}{\sqrt{3}}$

(3) $\cos\left(-\frac{50\pi}{3}\right)$

$= \cos\left(\frac{50\pi}{3}\right)$ ($\because$ cos is an even function)

$= \cos\left(\frac{51\pi - \pi}{3}\right)$

$= \cos\left(17\pi - \frac{\pi}{3}\right)$

$= -\cos\left(\frac{\pi}{3}\right)$

($\because$ $17\pi - \frac{\pi}{3}$ is in the second quadrant)

$= -\frac{1}{2}$

(4) sec690°

➠ $= \sec(720° - 30°)$

$= \sec 30°$ ($\because$ second quadrant)

$= \frac{2}{\sqrt{3}}$

(5) $\text{cosec}\frac{15\pi}{4}$

➠ $= \text{cosec}\left(4\pi - \frac{\pi}{4}\right)$

$= -\text{cosec}\frac{\pi}{4}$ ($\because$ fourth quadrant)

$= -\sqrt{2}$

(6) $\cot\left(-\frac{7\pi}{3}\right)$

➠ $= -\cot\frac{7\pi}{3}$ ($\because$ $\cot(-\theta) = -\cot\theta$)

➠ $= -\cot\left(2\pi + \frac{\pi}{3}\right)$ ($\because$ first quadrant)

$= -\cot\frac{\pi}{3}$

$= -\frac{1}{\sqrt{3}}$

Do It yourself

Evaluate : **(Based on Exercise 4.1, Example 1)**

(1) cos120° **(Ans. $-\frac{1}{2}$)**

(2) $\tan\left(\frac{-19\pi}{4}\right)$ **(Ans. 1)**

(3) $\sin\frac{5\pi}{4}$ **(Ans. $-\frac{1}{\sqrt{2}}$)**

(4) $\sec\left(\frac{37\pi}{6}\right)$ **(Ans. $\frac{2}{\sqrt{3}}$)**

(5) cosec 225° **(Ans. $-\sqrt{2}$)**

(6) $\cot\frac{15\pi}{4}$ **(Ans. -1)**

(7) $\cos\left(-\frac{11\pi}{3}\right)$ (Ans.

(8) $\sin\left(\frac{19\pi}{6}\right)$ (Ans.

(9) cosec 690° (Ans.

(10) sin150° (Ans.

❖ **Prove : (2 to 11)**

2. $\cos\left(\frac{\pi}{2}+\theta\right)\ \sec(-\theta)\ \tan(\pi-\theta) + \sec(2\pi+\theta)\cdot\sin(\pi+\theta)\cdot\cot\left(\frac{\pi}{2}-\theta\right) = 0$

➠ L.H.S. $= \cos\left(\frac{\pi}{2}+\theta\right)\cdot\sec(-\theta)\cdot\tan(\pi-\theta) + \sec(2\pi+\theta)\cdot\sin(\pi+\theta)\cdot\cot\left(\frac{\pi}{2}-\theta\right)$

$= -\sin\theta\cdot\sec\theta\cdot(-\tan\theta) + \sec\theta\cdot -\sin\theta\cdot\tan\theta$

$= \sin\theta\cdot\sec\theta\ \tan\theta - \sec\theta\cdot\sin\theta\cdot\tan\theta$

$= 0$

= R.H.S.

3. $\frac{\sin(\pi-\theta)}{\sin(\pi+\theta)}\cdot\frac{\text{cosec}(\pi+\theta)}{\text{cosec}(-\pi+\theta)}\cdot\frac{\text{cosec}(2\pi+\theta)}{\sin(3\pi-\theta)} = -\text{cosec}^2\theta$ **[Similar April 2015]**

➠ L.H.S. $= \frac{\sin(\pi-\theta)}{\sin(\pi+\theta)}\cdot\frac{\text{cosec}(\pi+\theta)}{\text{cosec}(-\pi+\theta)}\cdot\frac{\text{cosec}(2\pi+\theta)}{\sin(3\pi-\theta)}$

$= \frac{\sin\theta}{-\sin\theta}\cdot\frac{-\text{cosec}\,\theta}{\text{cosec}\,[-(\pi-\theta)]}\cdot\frac{\text{cosec}\,\theta}{\sin\theta}$

$= (-1)\cdot\frac{-\text{cosec}\,\theta}{-\text{cosec}\,(\pi-\theta)}\cdot\text{cosec}^2\theta$

$= (-1)\cdot\frac{-\text{cosec}\,\theta}{-\text{cosec}\,\theta}\cdot\text{cosec}^2\theta$

$= -\text{cosec}^2\theta$

= R.H.S.

4. $\frac{\sin(-\theta)\cdot\tan\left(\frac{\pi}{2}-\theta\right)\cdot\sin(\pi-\theta)\cdot\sec\left(\frac{3\pi}{2}+\theta\right)}{\sin(\pi+\theta)\cdot\cos\left(\frac{3\pi}{2}-\theta\right)\cdot\text{cosec}\,(\pi-\theta)\cdot\cot(2\pi-\theta)} = 1$

➠ L.H.S.

$= \frac{\sin(-\theta)\tan\left(\frac{\pi}{2}-\theta\right)\sin(\pi-\theta)\sec\left(\frac{3\pi}{2}+\theta\right)}{\sin(\pi+\theta)\cos\left(\frac{3\pi}{2}-\theta\right)\text{cosec}\,(\pi-\theta)\cot(2\pi-\theta)}$

$= \frac{-\sin\theta\cdot\cot\theta\cdot\sin\theta\cdot\text{cosec}\theta}{-\sin\theta\cdot(-\sin\theta)\cdot\text{cosec}\theta\cdot(-\cot\theta)}$

$= 1 =$ R.H.S.

5. $\sin(n+1)A\cdot\cos(n+2)A - \cos(n+1)A\cdot\sin(n+2)A = -\sin A$

➠ L.H.S. $= \sin(n+1)A\cos(n+2)A - \cos(n+1)A\sin(n+2)A$

$= \sin[(n+1)A-(n+2)A]$

$= \sin[nA + A - nA - 2A]$

$= \sin(-A)$

$= -\sin A =$ R.H.S.

6. $\sin^2(40°+\theta) + \sin^2(50°-\theta) = 1$

➠ L.H.S. $= \sin^2(40+\theta) + \sin^2(90-40-\theta)$

$= \sin^2(40+\theta) + \sin^2(90-(40+\theta))$

$= \sin^2(40+\theta) + \cos^2(40+\theta)$ ($\because \sin(90-\theta) = \cos\theta$)

$= 1$ ($\because$ foundamental identity)

= R.H.S.

7. $\frac{\cot 333° - \cos 567°}{\tan 297° + \sin 477°} = 1$

➠ L.H.S. $= \frac{\cot 333° - \cos 567°}{\tan 297° + \sin 477°}$

$= \frac{\cot(360°-27°) - \cos(540+27°)}{\tan(270+27) + \sin(450+27°)}$

$= \frac{-\cot 27 - (-\cos 27)}{-\cot 27 + \cos 27}$

$= \frac{-\cot 27 + \cos 27}{-\cot 27 + \cos 27}$

$= 1 =$ R.H.S.

8. $\frac{\sec^2 129° - \text{cosec}^2 31°}{\text{cosec}\ 39° - \sec 121°} = \text{cosec}39° - \sec 5$ **[March 201**

➠ L.H.S. $= \frac{\sec^2 129° - \text{cosec}^2 31°}{\text{cosec}\ 39° - \sec 121°}$

$= \dfrac{\sec^2(90+39)^\circ - \csc^2(90-59)^\circ}{\csc 39^\circ - \sec(180-59)^\circ}$

$= \dfrac{\csc^2 39 - \sec^2 59^\circ}{\csc 39 + \sec 59^\circ}$

$= \dfrac{(\csc 39^\circ - \sec 59^\circ)(\csc 39^\circ + \sec 59^\circ)}{\csc 39^\circ - \sec 59^\circ}$

$= \csc 39^\circ - \sec 59^\circ$

= R.H.S.

cos(A + B + C) = cosA cosB cosC − sinA sinB · cosC − sinA cosB sinC − cosA sinB sinC

L.H.S. $= \cos(A+B+C)$

$= \cos[A + (B+C)]$

$= \cos A \cos(B+C) - \sin A \sin(B+C)$

$= \cos A\,[\cos B \cos C - \sin B \sin C] - \sin A[\sin B \cos C + \cos B \sin C]$

$= \cos A \cos B \cos C - \sin A \cdot \sin B \cdot \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

= R.H.S.

10. $\sin\alpha \cdot \sin(\beta-\gamma) + \sin\beta \cdot \sin(\gamma-\alpha) + \sin\gamma \cdot \sin(\alpha-\beta) = 0$

➠ L.H.S. $= \sin\alpha \sin(\beta-\gamma) + \sin\beta \sin(\gamma-\alpha) + \sin\gamma \sin(\alpha-\beta)$

$= \sin\alpha[\sin\beta\cos\gamma - \cos\beta\sin\gamma] + \sin\beta[\sin\gamma\cos\alpha - \cos\gamma \sin\alpha] + \sin\gamma[\sin\alpha \cos\beta - \cos\alpha \sin\beta]$

$= \sin\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta\sin\gamma + \cos\alpha\sin\beta\sin\gamma - \sin\alpha \sin\beta \cos\gamma + \sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\sin\gamma$

$= 0$ = R.H.S.

11. $(\sin\alpha - \cos\alpha)\cdot(\sin\beta + \cos\beta) = \sin(\alpha-\beta) - \cos(\alpha+\beta)$

➠ L.H.S. $= (\sin\alpha - \cos\alpha) \cdot (\sin\beta + \cos\beta)$

$= \sin\alpha(\sin\beta + \cos\beta) - \cos\alpha(\sin\beta + \cos\beta)$

$= \sin\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta - \cos\alpha\cos\beta$

$= \sin\alpha \cos\beta - \cos\alpha\sin\beta + \sin\alpha\sin\beta - \cos\alpha\cos\beta$

$= [\sin\alpha\cos\beta - \cos\alpha\sin\beta] - [\cos\alpha\cos\beta - \sin\alpha\sin\beta]$

$= \sin(\alpha - \beta) - \cos(\alpha + \beta)$

= R.H.S.

Do It yourself

Prove : **(Based on Exercise 4.1, Example 2 to 11)**

(1) $\sin\dfrac{10\pi}{3}\tan\left(\dfrac{-23\pi}{6}\right) + \sec\dfrac{14\pi}{3}\cot\left(\dfrac{-21\pi}{4}\right) = \dfrac{3}{2}$

(2) $\cot\dfrac{\pi}{20}\ \cot\dfrac{3\pi}{20}\ \cot\dfrac{5\pi}{20}\ \cot\dfrac{7\pi}{20}\ \cot\dfrac{9\pi}{20} = 1$

(3) $\cos\left(\dfrac{3\pi}{4}+x\right) - \cos\left(\dfrac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

(4) $\cos^2 45^\circ - \sin^2 15^\circ = \dfrac{\sqrt{3}}{4}$

(5) $\dfrac{\sin\left(\dfrac{\pi}{2}+A\right)\cos(\pi-A)\tan(\pi+A)\sec(2\pi-A)}{\sec(16\pi-A)\tan(15\pi-A)\cos(14\pi+A)\sin\left(\dfrac{27\pi}{2}-A\right)} = -1$

(6) $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 190^\circ + \sin 200^\circ + \sin 220^\circ = 0$

(7) $\sin^2\dfrac{\pi}{4} + \sin^2\dfrac{3\pi}{4} - \sin^2\dfrac{7\pi}{4} - \sin^2\dfrac{5\pi}{4} = 0$

(8) $\cot\dfrac{\pi}{10}\cdot\cot\dfrac{2\pi}{10}\cdot\cot\dfrac{3\pi}{10}\cdot\cot\dfrac{4\pi}{10} = 1$

12. For ΔABC prove following results :

(1) $\sin(B + C) = \sin A$

➠ $\sin(B + C) = \sin(180° - A)$ $(\because A + B + C = 180°)$

$= \sin A$

(2) $\cos(A + B) = -\cos C$

➠ $\cos(A + B) = \cos(180° - C)$ $(\because A + B + C = 180°)$

$= -\cos C$

(3) $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$

➠ $\sin\left(\frac{B+C}{2}\right) = \sin\left(\frac{180-A}{2}\right)$ $(\because A + B + C = 180°)$

$= \sin\left(90 - \frac{A}{2}\right)$

$= \cos\frac{A}{2}$

(4) $\tan(A - B - C) = \tan 2A$.

➠ $\tan(A - B - C) = \tan[A - (B + C)]$

$= \tan[A - (180 - A)]$ $(\because A + B + C = 180°)$

$= \tan[A - 180 + A]$

$= \tan[-(180 - 2A)]$

$= -\tan(180 - 2A)$ $(\because \tan(-\theta) = -\tan\theta)$

$= -[-\tan 2A]$

$= \tan 2A$

(5) $\frac{\sin(B + C) \cdot \cos(B + C) \sin\frac{A}{2} \cos\frac{A}{2}}{\sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B+C}{2}\right) \sin(\pi + A) \cos(2\pi - A)}$ $= 1.$

➠ L.H.S.

$= \frac{\sin(B + C)\cos(B + C)\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right)\sin(\pi + A)\cos(2\pi - A)}$

$= \frac{\sin(180° - A)\cos(180 - A) \cdot \sin\frac{A}{2}\cos\frac{A}{2}}{\sin\left(\frac{180-A}{2}\right)\cos\left(\frac{180-A}{2}\right)(-\sin A)\cos A}$

$= \frac{\sin A \cdot (-\cos A) \cdot \sin\frac{A}{2}\cos\frac{A}{2}}{\sin\left(90 - \frac{A}{2}\right)\cos\left(90 - \frac{A}{2}\right)(-\sin A)(\cos A)}$

$= \frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}\sin\frac{A}{2}}$

$= 1 =$ R.H.S.

(6) If $\cos A = \cos B \cos C$, then prove that $2\cot B \cot C = 1$.

➠ Given $\cos A = \cos B \cos C$

$\therefore \cos(180° - (B + C)) = \cos B \cos C$

$\therefore -\cos(B + C) = \cos B \cos C$

$\therefore -\cos B \cos C + \sin B \sin C = \cos B \cos C$

$\therefore \sin B \sin C = \cos B \cos C + \cos B \cos C$

$\therefore 2\cos B \cos C = \sin B \sin C$

$\therefore \frac{2\cos B\cos C}{\sin B \cdot \sin C} = 1$

$\therefore 2\cot B \cot C = 1$

Do It yourself

❖ **For ΔABC prove.** **(Based on Exercise 4.1, Example 12)**

(1) $\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$

(2) $\sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B+C}{2}\right) \sin A \cos A + \sin\frac{A}{2} \cos\frac{A}{2} \sin(B + C) \cdot \cos(B + C) = 0$

(3) If ΔABC is a right angled triangle then $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(4) For ΔABC prove that,

(i) $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$ **(ii)** $\tan\frac{A+B}{2} = \cot\frac{C}{2}$.

13. For a convex quadrilateral ABCD prove that,

(1) sin(A + B) + sin(C + D) = sin(B + C) + sin(A + D)

⇒ A + B + C + D = 2π

L.H.S. = sin(A + B) + sin(C + D)

= sin(2π − (C + D)) + sin(C + D)

= −sin(C + D) + sin(C + D)

= 0

R.H.S. = sin(B + C) + sin(A + D)

= sin[2π − (A + D)] + sin(A + D)

= −sin(A + D) + sin(A + D)

= 0

L.H.S. = R.H.S.

(2) cot(A + B + C) + cotD = 0

⇒ A + B + C + D = 2π

L.H.S. = cot(A + B + C) + cotD

= cot(2π − D) + cot D

= −cot D + cot D

= 0 = R.H.S.

14. For cyclic quadrilateral ABCD prove that,

(1) cosA + cosB + cosC + cosD = 0

⇒ □ABCD is a cyclic quadrilateral

A + C = π and B + D = π

⇒ L.H.S. = cosA + cosB + cosC + cosD

= cosA + cosB + cos(π − A) + cos(π − B)

= cosA + cosB − cosA − cosB

= 0 = R.H.S.

(2) sinA + sinB = sinC + sinD

⇒ □ABCD is a cyclic quadrilateral.

A + C = π and B + D = π

⇒ L.H.S. = sinA + sinB

= sin(π − C) + sin(π − D)

= sinC + sin D

= R.H.S.

Do It yourself

Based on Exercise 4.1, Example 13 and 14)

1. For a convex quadrilateral ABCD prove that tan(A + B + C) + tanD = 0.
2. For cyclic quadrilateral ABCD Prove that tanA + cotB + tanC + cotD = 0.
3. In ΔABC cotB + cotC = 1 ⇒ sinB sinC = sinA.

15. If $\alpha - \beta = \frac{\pi}{6}$, then prove that $2\sin\alpha - \cos\beta = \sqrt{3}\sin\beta$.

⇒ $\alpha - \beta = \frac{\pi}{6}$

∴ $\alpha = \frac{\pi}{6} + \beta$

∴ $\sin\alpha = \sin\left(\frac{\pi}{6} + \beta\right)$

∴ $\sin\alpha = \sin\frac{\pi}{6}\cos\beta + \cos\frac{\pi}{6}\sin\beta$

∴ $\sin\alpha = \frac{1}{2}\cdot\cos\beta + \frac{\sqrt{3}}{2}\sin\beta$

∴ $2\sin\alpha = \cos\beta + \sqrt{3}\sin\beta$

∴ $2\sin\alpha - \cos\beta = \sqrt{3}\sin\beta$

16. If $\theta = \frac{19\pi}{4}$, then prove that $\cos^2\theta - \sin^2\theta - 2\tan\theta + \sec^2\theta - 4\cot^2\theta = 0$.

⇒ $\cos^2\theta - \sin^2\theta - 2\tan\theta + \sec^2\theta - 4\cot^2\theta$

$= \cos^2\frac{19\pi}{4} - \sin^2\frac{19\pi}{4} - 2\tan\frac{19\pi}{4} + \sec^2\frac{19\pi}{4} - 4\cot^2\frac{19\pi}{4}$

$= \cos^2\left(5\pi - \frac{\pi}{4}\right) - \sin^2\left(5\pi - \frac{\pi}{4}\right) - 2\tan\left(5\pi - \frac{\pi}{4}\right) + \sec^2\left(5\pi - \frac{\pi}{4}\right) - 4\cot^2\left(5\pi - \frac{\pi}{4}\right)$

$= \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4} - 2\left(-\tan\frac{\pi}{4}\right) + \sec^2\frac{\pi}{4} - 4\cot^2\frac{\pi}{4}$

$= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 + 2 + (\sqrt{2})^2 - 4$

$= \frac{1}{2} - \frac{1}{2} + 2 + 2 - 4$

$= 0 = $ R.H.S.

17. Evaluate :

(1) $\sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \sin^2\frac{5\pi}{12} + \sin^2\frac{7\pi}{12} + \sin^2\frac{9\pi}{12} + \sin^2\frac{11\pi}{12}$

➠ R.H.S. $= \sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \sin^2\frac{5\pi}{12} + \sin^2\frac{7\pi}{12} + \sin^2\frac{9\pi}{12} + \sin^2\frac{11\pi}{12}$

$= \sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \sin^2\left(\frac{6\pi-\pi}{12}\right) + \sin^2\left(\frac{6\pi+\pi}{12}\right) + \sin^2\left(\frac{6\pi+3\pi}{12}\right) + \sin^2\left(\frac{12\pi-\pi}{12}\right)$

$= \sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \sin^2\left(\frac{\pi}{2}-\frac{\pi}{12}\right) + \sin^2\left(\frac{\pi}{2}+\frac{\pi}{12}\right) + \sin^2\left(\frac{\pi}{2}+\frac{3\pi}{12}\right) + \sin^2\left(\pi-\frac{\pi}{12}\right)$

$= \sin^2\frac{\pi}{12} + \sin^2\frac{3\pi}{12} + \cos^2\frac{\pi}{12} + \cos^2\frac{\pi}{12} + \cos^2\frac{3\pi}{12} + \sin^2\frac{\pi}{12}$

$= 2\left(\sin^2\frac{\pi}{12} + \cos^2\frac{\pi}{12}\right) + \left(\sin^2\frac{3\pi}{12} + \cos^2\frac{3\pi}{12}\right)$

$= 2[1] + 1$

$= 3 = $ R.H.S.

(2) $\sin x + \sin(\pi + x) + \sin(2\pi + x) + \ldots$ **terms.**

➠ $\sin x + \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi +$ …

$+ \ldots$

$= \sin x - \sin x + \sin x - \sin x + \ldots\ 2n$

$= 0$

(3) $\cos x + \cos(\pi - x) + \cos(2\pi - x) + \cos(3\pi - x) + \ldots + (2n + 1)$ **terr…**

if $x =$ …

➠ $\cos x + \cos(\pi - x) + \cos(2\pi - x) + \cos(3\pi - x) + \ldots\ (2n + 1)$ te…

➠ $= \cos x - \cos x + \cos x - \cos x + \ldots\ (2n + 1)$ te…

($\because$ 2n term becomes zero and one term increas…

$= + \cos x$

$= + \cos\frac{\pi}{3} = +\frac{1}{2}$

(4) $\cot\frac{\pi}{20}\ \cot\frac{3\pi}{20}\ \cot\frac{5\pi}{20}\ \cot\frac{7\pi}{20}\ \cot\frac{9\pi}{20}$

➠ $\cot\frac{\pi}{20}\ \cot\frac{3\pi}{20}\ \cot\frac{5\pi}{20}\ \cot\frac{7\pi}{20}\ \cot\frac{9\pi}{20}$

$= \cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cdot\cot\frac{\pi}{4}\ \cot\left(\frac{\pi}{2}-\frac{3\pi}{20}\right)\cdot\cot\left(\frac{\pi}{2}-\ldots\right.$

$= \cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cdot 1\cdot\tan\frac{3\pi}{20}\cdot\tan\frac{\pi}{20}$

$\left(\because \cot\frac{\pi}{4} = 1\right)$

$= \cot\frac{\pi}{20}\ \tan\frac{\pi}{20}\cdot\cot\frac{3\pi}{20}\ \tan\frac{3\pi}{20}$

$= 1$

Do It yourself

❖ **Evaluate (1 to 3)** **(Based on Exercise 4.1, Example 17)**

(1) $\cos^2 37\frac{1}{2}^\circ - \cos^2 82\frac{1}{2}^\circ$ **(Ans.** $\frac{\sqrt{6}}{4}$**)**

(2) $\sin\frac{\pi}{14}\ \sin\frac{3\pi}{14}\ \sin\frac{5\pi}{14}\ \sin\frac{7\pi}{14}\ \sin\frac{9\pi}{14}\ \sin\frac{11\pi}{14}\ \sin\frac{13\pi}{14}$ **(Ans.** $\frac{1}{64}$**)**

(3) $\cos\frac{\pi}{7}\ \cos\frac{2\pi}{7}\ \cos\frac{4\pi}{7}$ **(Ans.** $-\frac{1}{8}$**)**

❖ **Prove : (Based on Exercise 4.1, Example 15 and 16)**

(4) (i) $\sec\left(\frac{3\pi}{2}-\theta\right)\ \sec\left(\theta-\frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2}+\theta\right)\ \tan\left(\theta-\frac{3\pi}{2}\right) = -1$

(ii) $\left\{1+\cot\theta-\sec\left(\frac{\pi}{2}+\theta\right)\right\}\left\{1+\cot\theta+\sec\left(\frac{\pi}{2}+\theta\right)\right\} = 2\cot\theta.$

Determine whether each of the following is positive or negative.

(1) sin155 + cos155

$\Rightarrow$ sin155 + cos155

= sin(180 − 25) + cos(90 + 65)

= sin25 − sin65

Since 25° < 65° is in the first quadrant and sine is an increasing (↑) function in the first quadrant.

∴ sin25° < sin65°

∴ sin25° − sin65 < 0

∴ This sign of the given function is negative (−ve)

(2) $\tan\frac{6\pi}{7} + \cot\left(-\frac{6\pi}{7}\right)$ **[April 2012]**

$\Rightarrow$ $\tan\frac{6\pi}{7} + \cot\left(-\frac{6\pi}{7}\right)$

$= \tan\frac{6\pi}{7} - \cot\frac{6\pi}{7}$

$= \tan\frac{12\pi}{14} - \cot\frac{12\pi}{14}$

$= \tan\left(\pi - \frac{2\pi}{14}\right) - \cot\left(\frac{\pi}{2} + \frac{5\pi}{14}\right)$

$= -\tan\frac{2\pi}{14} + \tan\frac{5\pi}{14}$

Now $\frac{2\pi}{14} < \frac{5\pi}{14}$, tan is an increasing function in the first quadrant.

$\therefore \tan\frac{2\pi}{14} < \tan\frac{5\pi}{14}$

$\therefore \tan\frac{5\pi}{14} - \tan\frac{2\pi}{14} > 0$

∴ The given number is positive.

(3) tan111° − cot111°

$\Rightarrow$ tan111° − cot111°

= tan(180 − 69) − cot(90 + 21)

= −tan69 + tan21

Now 69 > 21, tan is an increasing function in the first quadrant.

∴ tan69 > tan21

∴ tan21 − tan69 < 0

∴ The given number is negative.

(4) $\text{cosec}\frac{7\pi}{12} + \sec\frac{7\pi}{12}$

$\Rightarrow$ $\text{cosec}\frac{7\pi}{12} + \sec\frac{7\pi}{12}$

$= \text{cosec}\left(\pi - \frac{5\pi}{12}\right) + \sec\left(\frac{\pi}{2} + \frac{\pi}{12}\right)$

$\therefore \text{cosec}\frac{5\pi}{12} - \text{cosec}\frac{\pi}{12}$

Now $\frac{5\pi}{12} > \frac{\pi}{12}$ and in first quadrant sine is an increasing function and hence cosec is a decreasing function.

$\therefore \text{cosec}\frac{5\pi}{12} < \text{cosec}\frac{\pi}{12}$

$\therefore \text{cosec}\frac{5\pi}{12} - \text{cosec}\frac{\pi}{12} < 0$

∴ The given number is negative.

Do It yourself

Determine whether the following number are positive (+ve) or negative (-ve)

(Based on Exercise 4.1, Example 18)

(1) tan175° − cot175° **(Ans. Positive)**

(2) sin107° + cos107° **(Ans. Positive)**

(3) $\text{cosec}\frac{17\pi}{12} - \sec\frac{7\pi}{12}$ **(Ans. Positive)**

(4) sin144° − cos144° **(Ans. Positive)**

(5) sec200° − cosec200° **(Ans. Negative)**

(6) sin44° − cos33° **(Ans. Negative)**

(7) $\tan\frac{47\pi}{30} - \cot\frac{47\pi}{30}$ **(Ans. Negative)**

(8) tan111° + tan222° **(Ans. Negative)**

(9) Evaluate :
sinθ + sin(π + θ) + sin(2π + θ) + sin(3π + θ) + ... to $2n^{th}$ term. ($n \in N$) **(Ans. 0)**

(10) Prove that, sin(A + B + C) = sinAcosBcosC + cosAsinB cosC + cosAcosBsinC − sinAsinBsinC.

19. If $\tan\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$, then find the value of $\dfrac{\sin(\pi-\theta)+\tan(\pi+\theta)+\tan(4\pi-\theta)}{\sin\left(\frac{3\pi}{2}+\theta\right)+\cos\left(\frac{5\pi}{2}-\theta\right)}$.

➡ $\tan\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$

$1+\tan^2\theta = \sec^2\theta$

$\therefore \sec^2\theta = 1 + \left(-\frac{3}{4}\right)^2$

$= 1 + \frac{9}{16}$

$= \frac{25}{16}$

$\therefore \sec\theta = -\frac{5}{4}$ ($\because$ θ is in the second quadrant)

$\therefore \cos\theta = -\frac{4}{5}$

$\sin^2\theta = 1 - \cos^2\theta$

$= 1 - \left(-\frac{4}{5}\right)^2$

$= 1 - \frac{16}{25} = \frac{9}{25}$

$\therefore \sin\theta = \frac{3}{5}$ ($\because$ θ is in the second quadrant)

Now $\dfrac{\sin(\pi-\theta)+\tan(\pi+\theta)+\tan(4\pi-\theta)}{\sin\left(\frac{3\pi}{2}+\theta\right)+\cos\left(\frac{5\pi}{2}-\theta\right)}$

$= \dfrac{\sin\theta+\tan\theta-\tan\theta}{-\cos\theta+\sin\theta}$

$= \dfrac{\sin\theta}{\sin\theta-\cos\theta}$

$= \dfrac{\frac{3}{5}}{\frac{3}{5}-\left(-\frac{4}{5}\right)}$

$= \dfrac{\frac{3}{5}}{\frac{3}{5}+\frac{4}{5}}$

$= \dfrac{\frac{3}{5}}{\frac{7}{5}} = \frac{3}{7}$

20. Prove that $\sin(n\pi + (-1)^n\theta) = \sin\theta$, for $n \in N$.

(1) If n is an even number then

let $n = 2m$, $m \in N$

$\sin[n\pi + (-1)^n\theta]$

$= \sin[2m\pi + (1)^{2m}\theta]$

$= \sin[2m\pi + \theta]$ ($\because (-1)^{2m} = 1$)

$= \sin\theta$ ($\because 2m\pi + \theta$ expresses the first quadrant)

(2) If n is an odd number then,

let $n = 2m+1$, $m \in N$

$\sin[n\pi + (-1)^n\theta]$

$= \sin[(2m+1)\pi + (-1)^{2m+1}\theta]$

$= \sin[(2m\pi+1)\pi - \theta]$ ($\because (-1)^{2m+1} = -1$)

$= \sin\theta$ ($\because (2m+1)\pi - \theta$ expresses the second quadrant)

Do it yourself

❖ **(Based on Exercise 4.1, Example 19)**

(1) If $\tan\theta = \frac{3}{4}$, $0 < \theta < \frac{\pi}{2}$ then find the value of $\dfrac{\sin(\pi-\theta)+\tan(\pi+\theta)+\tan(4\pi-\theta)}{\sin\left(\frac{3\pi}{2}+\theta\right)+\cos\left(\frac{5\pi}{2}-\theta\right)}$ **(Ans. -3)**

(2) Prove : $\cos^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\cos A$.

(3) Prove :

(i) $\sin A \sin(B-C) + \sin B \sin(C-A) + \sin(\sin(A-B)) = 0$

(ii) $\cos A \sin(B-C) + \cos B \sin(C-A) + \cos C \sin(A-B) = 0$

(4) Prove : $2\sin\left(\alpha - \frac{\pi}{6}\right) = \sqrt{3}\sin\alpha - \cos\alpha$.

Exercise 4.2 [Textbook Page No. 78]

[illegible]

[illegible] $- \sin^2 7\frac{1}{2}^\circ$

[illegible] $- \sin^2 7\frac{1}{2}^\circ$

[illegible] $\cdot \sin\left(37\frac{1}{2} - 7\frac{1}{2}\right)^\circ$

[illegible] $\sin 30^\circ$

[illegible] $= \frac{1}{2\sqrt{2}}$

[illegible] $- \cos^2\left(7\frac{1}{2}^\circ\right)$

[illegible] $- \cos^2\left(7\frac{1}{2}^\circ\right)$

[illegible] $\left(7\frac{1}{2}^\circ\right) - \sin^2\left(52\frac{1}{2}^\circ\right)\}$

[illegible] $\left(7\frac{1}{2}^\circ + 52\frac{1}{2}^\circ\right)\cos\left(7\frac{1}{2}^\circ - 52\frac{1}{2}^\circ\right)\}$

[illegible] $\cos^2\alpha - \sin^2\alpha = \cos(\alpha + \beta)\cos(\alpha - \beta))$

$= -(\cos(60^\circ)\cos(-45^\circ))$

$= -\cos 60^\circ \cos 45^\circ$

$= -\frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

($\because \cos(-\theta) = \cos\theta$, cos is an even function.)

$= -\frac{1}{2\sqrt{2}}$

(3) $\cos^2 37\frac{1}{2}^\circ - \sin^2 37\frac{1}{2}^\circ$

➠ $\cos^2 37\frac{1}{2}^\circ - \sin^2 37\frac{1}{2}^\circ$

$= \cos\left(37\frac{1}{2}^\circ + 37\frac{1}{2}^\circ\right) \cdot \cos\left(37\frac{1}{2}^\circ - 37\frac{1}{2}^\circ\right)$

$= \cos(75^\circ)\cos 0^\circ$

$= \cos(45^\circ + 30^\circ)$ (1)

$= \cos 45 \cos 30^\circ - \sin 45 \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

Do It yourself

(Based on Exercise 4.2, Example 1)

Evaluate :

(1) $\cos^2 45^\circ - \sin^2 15^\circ$ **(Ans. $\frac{\sqrt{3}}{4}$)**

(2) $\frac{\cos^2 33^\circ - \cos^2 57^\circ}{\sin^2 \frac{21}{2}^\circ - \sin^2 \frac{69}{2}^\circ}$ **(Ans. $-\sqrt{2}$)**

[illegible] Prove that : $\sin^2 A + \sin^2 B + \cos^2(A + B) - 2\sin A \sin B \cdot \cos(A + B) = 1$.

L.H.S. $= \sin^2 A + \sin^2 B + \cos^2(A + B) + 2\sin A \sin B \cdot \cos(A + B)$

$= 1 - \cos^2 A + \sin^2 B + \cos(A + B)[\cos(A + B) + 2\sin A \sin B]$

$= 1 - [\cos^2 A - \sin^2 B] + \cos(A + B)[\cos A \cos B - \sin A \sin B + 2\sin A \sin B]$

$= 1 - \cos(A + B)\cos(A - B) + \cos(A + B)[\cos A \cos B + \sin A \sin B]$

$= 1 - \cos(A + B)\cos(A - B) + \cos(A + B)\cos(A - B)$

$= 1 =$ R.H.S.

3.(1) If $\cos A = \frac{1}{7}$, $\cos B = \frac{13}{14}$ and $0 < A, B < \frac{\pi}{2}$, then prove that $A - B = \frac{\pi}{3}$.

➡ Here $0 < A < \frac{\pi}{2}$ .....(i)

$0 < B < \frac{\pi}{2}$

$\therefore 0 > -B > -\frac{\pi}{2}$

$\therefore -\frac{\pi}{2} < -B < 0$ .....(ii)

Taking (i) + (ii),

$-\frac{\pi}{2} < A - B < \frac{\pi}{2}$

Thus (A – B) is in either first or fourth quadrant.

Here to prove $A - B = \frac{\pi}{3}$,

$\sin(A - B) = \frac{\sqrt{3}}{2}$ is to be proved.

$\because \cos(A - B)$ is positive in the both the quadrants.

Now $\cos A = \frac{1}{7}$, $0 < A < \frac{\pi}{2}$

$\therefore \sin A = \sqrt{1 - \cos^2 A}$

$= \sqrt{1 - \frac{1}{49}} = \sqrt{\frac{48}{49}}$

$\therefore \sin A = \frac{4\sqrt{3}}{7}$ .....(I)

Also $\cos B = \frac{13}{14}$

$\sin B = \sqrt{1 - \cos^2 B}$

$= \sqrt{1 - \frac{169}{196}}$

$= \sqrt{\frac{27}{196}} = \frac{3\sqrt{3}}{14}$

Now $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$= \frac{4\sqrt{3}}{7} \cdot \left(\frac{13}{14}\right) - \frac{1}{7}\left(\frac{3\sqrt{3}}{14}\right)$

$= \frac{(\sqrt{3})}{7(14)} (4(13) - 3)$

$= \frac{(\sqrt{3})(49)}{7(14)} = \frac{\sqrt{3}}{2}$

$\therefore \sin(A - B) = \frac{\sqrt{3}}{2}$

$= \sin\frac{\pi}{3}$

$\therefore A - B = \frac{\pi}{3}$ is proved.

(2) If $\sin A = \frac{1}{\sqrt{5}}$, $\cos B = \frac{3}{\sqrt{10}}$ and $0 < A, B < \frac{\pi}{2}$, then prove that $A + B = \frac{\pi}{4}$.

➡ $\sin A = \frac{1}{\sqrt{5}}$

$\cos^2 A = 1 - \sin^2 A$

$= 1 - \left(\frac{1}{\sqrt{5}}\right)^2$

$= 1 - \frac{1}{5}$

$= \frac{4}{5}$

$\therefore \cos A = \frac{2}{\sqrt{5}}$

$\left(\because 0 < A < \frac{\pi}{2}\right)$

$\cos B = \frac{3}{\sqrt{10}}$

$\sin^2 B = 1 - \cos^2 B$

$= 1 - \left(\frac{3}{\sqrt{10}}\right)^2$

$= 1 - \frac{9}{10}$

$= \frac{1}{10}$

$\therefore \sin B = \frac{1}{\sqrt{10}}$

$\left(\because 0 < B < \frac{\pi}{2}\right)$

Now $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}}$

$= \frac{3}{\sqrt{5} \cdot \sqrt{10}} + \frac{2}{\sqrt{5} \cdot \sqrt{10}}$

$= \frac{3 + 2}{\sqrt{5} \cdot \sqrt{10}}$

$= \frac{5}{\sqrt{5} \cdot \sqrt{10}}$

$= \frac{\sqrt{5}}{\sqrt{10}}$

$= \sqrt{\frac{5}{10}}$

$= \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\therefore A + B = \frac{\pi}{4}$

(1) Find the quadrant of $P(\alpha - \beta)$, if $\cos\alpha = \frac{4}{5}$, $\cos\beta = \frac{12}{13}$, $\frac{3\pi}{2} < \alpha, \beta < 2\pi$.

[March 2016]

Here $\frac{3\pi}{2} < \alpha,\ \beta < 2\pi$.

$\therefore\ \frac{3\pi}{2} < \alpha < 2\pi$ .....(1)

and $\frac{3\pi}{2} < \beta < 2\pi$

$-\frac{3\pi}{2} > -\beta\ .\ -2\pi$

$\therefore\ -2\pi < -\beta < -\frac{3\pi}{2}$ .....(2)

Taking (1) + (2),

$\frac{3\pi}{2} - 2\pi < \alpha - \beta < 2\pi - \frac{3\pi}{2}$

$-\frac{\pi}{2} < \alpha - \beta < \frac{\pi}{2}$

$\therefore\ \alpha - \beta$ is in the first or fourth quadrant.

Now $P(\alpha - \beta) = (\cos(\alpha - \beta), \sin(\alpha - \beta))$.

But $\cos(\alpha - \beta)$ is positive in both the quadrants hence to determine the answer we have to veryfy the sign of $\sin(\alpha - \beta)$ also.

Here, $\cos\alpha = \frac{4}{5}$

$\sin\alpha = \sqrt{1 - \cos^2\alpha}$

$= \sqrt{1 - \frac{16}{25}}$

$= \sqrt{\frac{4}{5}}$

$= -\frac{3}{5}$ $\left(\because \frac{3\pi}{2} < \alpha < 2\pi\right)$

$\cos\beta = \frac{12}{13}$

$\sin\beta = \sqrt{1 - \sin^2\alpha}$

$= \sqrt{1 - \frac{144}{169}}$

$= \sqrt{\frac{25}{169}}$

$= -\frac{5}{13}$ $\left(\because \frac{3\pi}{2} < \beta < 2\pi\right)$

$\therefore\ \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= -\frac{3}{5}\left(\frac{12}{13}\right) - \frac{4}{5}\left(-\frac{5}{13}\right)$

$= \frac{-36 + 20}{65}$

$= -\frac{16}{65}$

Thus $\sin(\alpha - \beta)$ is negative hence $P(\alpha - \beta)$ is in the fourth quadrant.

(2) Find the quadrant of $P(\alpha + \beta)$, if $\cos\alpha = \frac{-5}{13}$, $\frac{\pi}{2} < \alpha < \pi$ and $\tan\beta = \frac{4}{3}$, $\pi < \beta < \frac{3\pi}{2}$.

➠ $\cos\alpha = \frac{-5}{13}$

$\sin^2\alpha = 1 - \cos^2\alpha$

$= 1 - \frac{25}{169}$

$= \frac{144}{169}$

$\therefore\ \sin\alpha = \frac{12}{13}$ $\left(\because \frac{\pi}{2} < \alpha < 2\pi\right)$

$\tan\beta = \frac{4}{3}$

$\sec^2\beta = 1 + \tan^2\beta$

$= 1 + \left(\frac{4}{3}\right)^2$

$= 1 + \frac{16}{9}$

$= \frac{25}{9}$

$\therefore\ \sec\beta = \frac{-5}{3}$ $\left(\because \pi < \beta < \frac{3\pi}{2}\right)$

$\therefore\ \cos\beta = \frac{-3}{5}$

$\therefore\ \sin\beta = \sqrt{1 - \cos^2\beta}$

$= \sqrt{1 - \frac{9}{25}}$

$= \sqrt{\frac{16}{25}}$

$= \frac{-4}{5}$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{12}{13}\right)\left(\frac{-3}{5}\right) + \left(\frac{-5}{13}\right)\left(\frac{-4}{5}\right)$

$= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65} < 0$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(\frac{-5}{13}\right)\left(\frac{-3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{-4}{5}\right)$

$= \frac{15}{65} + \frac{48}{65} = \frac{63}{65} > 0$

Thus $\sin(\alpha + \beta) < 0$ and $\cos(\alpha + \beta) > 0$.

$\therefore P(\alpha + \beta)$ is in the fourth quadrant.

5. If $\cot\alpha = \frac{1}{2}$, $\sec\beta = -\frac{5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$. Find the value of $\tan(\alpha + \beta)$ and find the quadrant of $P(\alpha + \beta)$.

➠ $\cos\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$

Now $1 + \tan^2\beta = \sec^2\beta$

$\therefore \tan^2\beta = -\sqrt{\sec^2\beta - 1}$

$= -\sqrt{\frac{25}{9} - 1}$

$= -\frac{4}{3} \quad \left(\because \frac{\pi}{2} < \beta < \pi\right)$

Now $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{2 - \frac{4}{3}}{1 - (2)\left(-\frac{4}{3}\right)} = \frac{2}{11}$

Now $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$

$\Rightarrow \pi + \frac{\pi}{2} < \alpha + \beta < \frac{3\pi}{2} + \pi$

$\Rightarrow \frac{3\pi}{2} < \alpha + \beta < \frac{5\pi}{2}$

$\therefore P(\alpha + \beta)$ is in the first and fourth quadrant

but $\tan(\alpha + \beta) = \frac{2}{11} > 0$.

$\therefore P(\alpha + \beta)$ is in the first quadrant.

Do It yourself

❖ **(Based on Exercise 4.2, Example 4.5)**

(1) If $\sin\alpha = -\frac{3}{5}$, $\pi < \alpha < \frac{3\pi}{2}$ and $\cos\beta = \frac{9}{41}$, $\frac{3\pi}{2} < \beta < 2\pi$ then determine the quadrant of $P(\alpha + \beta)$.
(Ans. Second)

(2) If $\sin\alpha = -\frac{4}{5}$, $3\pi < \alpha < \frac{7\pi}{2}$ and $\tan\beta = -\frac{12}{5}$, $\frac{5\pi}{2} < \beta < 3\pi$ then determine the quadrant of $P(\alpha - \beta)$.
(Ans. Second)

(3) If $\cos\alpha = -\frac{24}{25}$ and $\tan\beta = \frac{9}{40}$ and $\frac{\pi}{2} < \alpha < 2\pi$ and $\pi < \beta < \frac{3\pi}{2}$ then determine the quadrant of $P(\alpha + \beta)$.
(Ans. Fourth)

(4) If $\sin\alpha = \frac{3}{5}$, $\frac{25\pi}{2} < \alpha < 13\pi$, $\cos\beta = -\frac{5}{13}$, $15\pi < \beta < \frac{31\pi}{2}$ then determine the situations of $P(\alpha - \beta)$ and $P(\alpha + \beta)$.
(Ans. $P(\alpha - \beta)$ Third, $P(\alpha + \beta)$ first)

6. Determine the range of :

(1) $7\sin\theta + 24\cos\theta$

➠ Comparing $f(\theta) = 24\cos\theta + 7\sin\theta$ with $a\cos\theta + b\sin\theta$

$a = 24,\ b = 7$

Now $r^2 = a^2 + b^2 = (24)^2 + (7)^2$

$= 576 + 49$

$= 625$

$\therefore r = 25$

$\therefore$ The range of $f(\theta)$, $[-r, r] = [-25, 25]$

$= \cos\theta + \sin\left(\theta - \frac{\pi}{6}\right) + 1$

$= \cos\theta + \sin\left(\theta - \frac{\pi}{6}\right) + 1$

$= \cos\theta + \sin\theta \cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6} + 1$

$= \cos\theta + \sin\theta \cdot \frac{\sqrt{3}}{2} - \cos\theta \cdot \frac{1}{2} + 1$

$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta + 1$

Now comparing $f(\theta) = \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$ with $a\cos\theta + b\sin\theta$,

$a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$

$\therefore r^2 = a^2 + b^2 = \frac{1}{4} + \frac{3}{4} = 1$

$\therefore r = 1$

$\therefore$ The range of $f(\theta) = \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$.

$[-r, r] = [-1, 1]$

i.e. $-1 \le \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \le 1$,

$\therefore -1 + 1 \le \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta + 1 \le 1 + 1$

$\therefore 0 \le \cos\theta + \sin\left(\theta - \frac{\pi}{6}\right) + 1 \le 2$

$\therefore$ The range of $\cos\theta + \sin\left(\theta - \frac{\pi}{6}\right) + 1$ is $[0, 2]$.

Do It yourself

(Based on Exercise 4.2, Example 6)

Find the range :

1. $\sqrt{3}\cos\theta - \sin\theta$ **(Ans. $[-2, 2]$)**
2. $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ **(Ans. $[-4, 10]$)**
3. $5\cos\theta + 12\sin\theta + 7$ **(Ans. $[-6, 20]$)**
4. $\sin\alpha + \cos\alpha$ **(Ans. $[-\sqrt{2}, \sqrt{2}]$)**
5. $3\cos\theta + 4\sin\theta + 5$ **(Ans. $[0, 10]$)**
6. $\sin\theta + \cos\left(\theta + \frac{\pi}{3}\right)$ **(Ans. $\left[-\frac{\sqrt{6}-\sqrt{2}}{2}, \frac{\sqrt{6}-\sqrt{2}}{2}\right]$)**
7. $\cos\alpha - \sqrt{3}\sin\alpha$ **(Ans. $[-2, 2]$)**

Prove that $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 7$ in $[0, 14]$.

$5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 7$

$= 5\cos\theta + 3\left[\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right] + 7$

$= 5\cos\theta + 3\left[\cos\theta \cdot \frac{1}{2} - \sin\theta \cdot \frac{\sqrt{3}}{2}\right] + 7$

$= 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 7$

$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 7$ ....(1)

Now comparing $f(\theta) = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$ with $a\cos\theta + b\sin\theta$,

$a = \frac{13}{2}$, $b = -\frac{3\sqrt{3}}{2}$

$\therefore r^2 = a^2 + b^2 = \left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2$

$= \frac{169}{4} + \frac{27}{4}$

$= \frac{196}{4} = 49$

$\therefore r = 7$

$\therefore$ The range of $f(\theta) = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

$= [-7, 7]$.

i.e. $-7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$.

$\therefore -7 + 7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 7 \le 7 + 7$

$\therefore 0 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 7 \le 14$

($\because$ From (1))

$\therefore$ The value of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 7$ is in $[0, 14]$.

8. Express $\sqrt{3}\sin\theta + \cos\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < 2\pi$.

➠ Let $f(\theta) = \sqrt{3}\sin\theta + \cos\theta$

$r = \sqrt{(\sqrt{3})^2 + (1)^2}$ = By multiplying and dividing by 2 $f(\theta)$

$f(\theta) = 2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$

$= 2\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)$

$= 2\cos\left(\theta - \frac{\pi}{3}\right)$

Now comparing $f(\theta)$ with $r\cos(\theta - \alpha)$,

$r = 2, \ \alpha = \frac{\pi}{3}$

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)$

9. $-\frac{\pi}{2} < \theta < 0$ and $\cos\alpha - \sqrt{3}\sin\alpha = r\cos(\alpha - \theta)$, find r and θ.

➠ Let $f(\theta) = \cos\alpha - \sqrt{3}\sin\alpha$.

$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$ Now dividing and multiplying f(θ) by 2,

$f(\theta) = 2\left(\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right)$

$= 2\left(\cos\frac{\pi}{3}\cos\alpha - \sin\frac{\pi}{3}\sin\alpha\right)$

$= 2\cos\left(\alpha + \frac{\pi}{3}\right)$

$= 2\cos\left(\alpha - \left(-\frac{\pi}{3}\right)\right)$

$= r\cos(\alpha - \theta)$

$\therefore r = 2$ and $\theta = -\frac{\pi}{3}$

10. Prove :

(1) $\tan\left(\frac{\pi}{3} - \alpha\right) = \frac{\sqrt{3}\cos\alpha - \sin\alpha}{\cos\alpha + \sqrt{3}\sin\alpha}$

➠ $\tan\left(\frac{\pi}{3} - \alpha\right) = \frac{\tan\frac{\pi}{3} - \tan\alpha}{1 + \tan\frac{\pi}{3}\tan\alpha}$

$= \frac{\sqrt{3} - \frac{\sin\alpha}{\cos\alpha}}{1 + \sqrt{3}\cdot\frac{\sin\alpha}{\cos\alpha}}$

$= \frac{\sqrt{3}\cos\alpha - \sin\alpha}{\cos\alpha + \sqrt{3}\sin\alpha}$

(2) $\tan 39° = \frac{\sqrt{3}\cos 21° - \sin 21°}{\cos 21° + \sqrt{3}\sin 21°}$

➠ $\tan 39° = \tan(60 - 21)°$

$= \frac{\tan 60° - \tan 21°}{1 + \tan 60° \tan 21°}$

$= \frac{\sqrt{3} - \frac{\sin 21°}{\cos 21°}}{1 + \sqrt{3}\cdot\frac{\sin 21°}{\cos 21°}}$

$= \frac{\sqrt{3}\cos 21° - \sin 21°}{\cos 21° + \sqrt{3}\sin 21°}$

(3) $\tan 3A \cdot \tan 2A \cdot \tan A = \tan 3A - \tan 2A - \tan A$

➠ $\tan 3A = \tan(2A + A)$

$\therefore \tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \cdot \tan A}$

$\tan 3A\,[1 - \tan 2A\,\tan A]$

$= \tan 2A + \tan A$

$\tan 3A - \tan 3A\,\tan 2A\tan A = \tan 2A + \tan A$

$\tan 3A - \tan 2A - \tan A = \tan 3A\,\tan 2A\,\tan A$

(4) $\cot A\cot 2A - \cot 2A\cdot\cot 3A - \cot 3A\cdot\cot A = 1$

$\cot 3A = \cot(2A + A)$

$\cot 3A = \dfrac{\cot 2A\cot A - 1}{\cot 2A + \cot A}$

$\cot 3A(\cot 2A + \cot A) = \cot 2A\,\cot A - 1$

$\cot 3A\cot 2A + \cot 3A\cot A = \cot 2A + \cot A - 1$

$\cot A\cot 2A - \cot 2A\,\cot 3A - \cot 3A\,\cot A = 1$

(5) $\tan 25^\circ\ \tan 15^\circ + \tan 15^\circ\tan 50^\circ + \tan 25^\circ\ \tan 50^\circ = 1$

➠ $\tan 75^\circ = \tan(50^\circ + 25^\circ)$

$\therefore\ \tan 75^\circ = \dfrac{\tan 50^\circ + \tan 25}{1 - \tan 50^\circ \tan 25}$

$\therefore\ \cot 15^\circ = \dfrac{\tan 50 + \tan 25}{1 - \tan 50\cdot\tan 25}$

$[\because\ \tan\theta = \cot(90 - \theta)]$

$\therefore\ \dfrac{1}{\tan 15} = \dfrac{\tan 50 + \tan 25}{1 - \tan 50\,\tan 25}$

$\therefore\ 1 - \tan 50\,\tan 25 = \tan 50\,\tan 15 + \tan 15\,\tan 25$

$\therefore\ \tan 25\,\tan 15 + \tan 50\,\tan 15 + \tan 50\,\tan 25 = 1$

Do It yourself

(Based on Exercise 4.2, Example 9)

(1) If $\cos\alpha - \sin\alpha = r\cos(\theta - \alpha)$ then find r and θ. where $0 < \theta < 2\pi$, $r > 0$. **(Ans.** $\sqrt{2}, \dfrac{7\pi}{4}$**)**

(2) If $\sin\theta + \cos\theta = r\cos(\theta - \alpha)$ then find r and α. where $0 < \alpha < \dfrac{\pi}{2}$, $r > 0$. **(Ans.** $\sqrt{2}, \dfrac{\pi}{4}$**)**

(3) If $\sqrt{3}\cos\alpha - \sin\alpha = r\cos(\alpha - \theta)$ then find r and θ. where $0 < \theta < 2\pi$, $r > 0$. **(Ans.** $2, \dfrac{11\pi}{6}$**)**

(4) **Prove : (Based on Exercise 4.2, Ex. No. 10)**

(1) $\cot\left(\dfrac{\pi}{6} + \theta\right) = \dfrac{\sqrt{3}\cos\theta - \sin\theta}{\cos\theta + \sqrt{3}\sin\theta}$

(2) $\tan 5^\circ\ \tan 35^\circ + \tan 35^\circ\ \tan 50^\circ + \tan 50^\circ\ \tan 5^\circ = 1$

(3) $\tan 25^\circ + \tan 75^\circ + \tan 80^\circ = \tan 25^\circ\cdot\tan 75^\circ\ \tan 80^\circ$

(4) $\cot 15^\circ + \cot 35^\circ + \cot 40^\circ = \cot 15^\circ\ \cot 35^\circ\ \cot 40^\circ$

(5) $\tan 36^\circ + \tan 9^\circ + \tan 36^\circ\ \tan 9^\circ = 1$

11. If $A + B = \dfrac{\pi}{4}$, then prove that,

(1) $(1 + \tan A)\ (1 + \tan B) = 2$

➠ $A + B = \dfrac{\pi}{4}$

$\therefore\ \tan(A + B) = \tan\dfrac{\pi}{4}$

$\therefore\ \dfrac{\tan A + \tan B}{1 - \tan A\,\tan B} = 1$

$\therefore\ \tan A + \tan B = 1 - \tan A\,\tan B$

$\therefore\ \tan A + \tan B + \tan A\,\tan B = 1$

$\therefore\ 1 + \tan A + \tan B + \tan A\,\tan B = 1 + 1$

$\therefore\ (1 + \tan A) + \tan B(1 + \tan B) = 2$

$\therefore\ (1 + \tan A)\,(1 + \tan B) = 2$

(2) $(\cot A - 1)(\cot B - 1) = 2$

➠ $A + B = \dfrac{\pi}{4}$

$\therefore\ \cot(A + B) = \cot\dfrac{\pi}{4}$

$\therefore\ \dfrac{\cot A\,\cot B - 1}{\cot A + \tan B} = 1$

$\therefore\ \cot A\,\cot B - 1 = \cot A + \cot B$

$\therefore\ \cot A\,\cot B - \cot A - \cot B = 1$

$\therefore\ \cot A\,\cot B - \cot A - \cot B + 1 = 1 + 1$

$\therefore\ \cot A(\cot B - 1) - 1(\cot B - 1) = 2$

$\therefore\ (\cot A - 1)(\cot B - 1) = 2$

12. (1) Prove $A + B = \frac{\pi}{2} \Rightarrow \tan A = \tan B + 2\tan(A - B)$

➠ $A + B = \frac{\pi}{2}$

$\therefore A = \frac{\pi}{2} - B$

$\therefore \tan A = \tan\left(\frac{\pi}{2} - B\right)$

$\therefore \tan A = \cot B$

$\therefore \tan A = \frac{1}{\tan B}$

$\therefore \tan A \tan B = 1$ ......(1)

Now $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\therefore \tan(A - B) = \frac{\tan A - \tan B}{1 + 1}$ (∵ From (1))

$\therefore 2\tan(A - B) = \tan A - \tan B$

$\therefore \tan A = \tan B + 2\tan(A - B)$

(2) Prove $\tan 65° = \tan 25° + 2\tan 40°$

➠ $65 = 40 + 25$

$\therefore \tan 65 = \tan(40 + 25)$

$\therefore \tan 65 = \frac{\tan 40 + \tan 25}{1 - \tan 40 \tan 25}$

$\therefore \tan 65 - \tan 65 \cdot \tan 40 \tan 25 = \tan 40 + \tan 25$

$\therefore \tan 65 - \tan(90 - 25)\tan 40 \tan 25 = \tan 40 + \tan 25$

$\therefore \tan 65 - \cot 25 \tan 40 \tan 25 = \tan 40 + \tan 25$

$\therefore \tan 65 - \tan 40 = \tan 40 + \tan 25$

$\therefore \tan 65 = 2\tan 40 + \tan 25$ which is the required result.

13. If $A + B + C = (2k + 1)\frac{\pi}{2}$, $k \in Z$, then prove that,

(1) $\tan A \tan B + \tan B \tan C + \tan C \cdot \tan A = 1$

➠ $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

$A + B + C = (2k + 1)\frac{\pi}{2}$, $k \in Z$

$\therefore A + B = (2k + 1)\frac{\pi}{2} - C$

… $+ B) = \tan\left((2k + 1)\frac{\pi}{2} - C\right)$

$= \cot C$

n … $\frac{\pi}{2} - C$ is in the first or third quadrant)

$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{\tan C}$

$\therefore \tan A \tan C + \tan B \tan C = 1 -$ …

$\therefore \tan A \tan B + \tan B \tan C + \tan A$ …

(2) $\cot A + \cot B + \cot C = \cot A \cot$ …

➠ $A + B + C = (2k + 1)\frac{\pi}{2}$

$\therefore A + B = (2k + 1)\frac{\pi}{2} - C$

$\therefore \cot(A + B) = \cot\left[(2k + 1)\frac{\pi}{2} - C\right]$

$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = \tan C = \frac{1}{\cot C}$

(∵ $(2k + 1)\frac{\pi}{2} - C$ expresses the first … quadrant)

$\therefore \cot A \cot B \cot C - \cot C = \cot A +$ …

$\therefore \cot A + \cot B + \cot C = \cot A \cot B$ …

14. If $A + B + C = k\pi$, $k \in Z$, then prove…

(1) $\tan A + \tan B + \tan C = \tan A \tan B$ …

➠ $A + B + C = k\pi$

$\therefore A + B = k\pi - C$

$\therefore \tan(A + B) = \tan(k\pi - C)$

$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

(∵ $(k\pi - C)$ expresses the second or fourth qua…

$\therefore \tan A + \tan B = -\tan C + \tan A \tan B \tan$ …

$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan$ …

(2) $\cot B \cot C + \cot C \cot A + \cot A \cot B$ …

➠ $A + B + C = k\pi$

$\therefore A + B = k\pi - C$

$\therefore \cot(A + B) = \cot(k\pi - C)$

$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$

(∵ $(k\pi - C)$ expresses the second or fourth qua…

$\therefore \cot A \cot B - 1 = -\cot A \cot C - \cot B$ …

$\therefore \cot A \cot B + \cot B \cot C + \cot A \cot C =$ …

15. If $\tan A = 3$, $\tan B = \frac{1}{2}$, $0 < A, B <$ … then prove that $A - B = \frac{\pi}{4}$.

➠ $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{3 - \frac{1}{2}}{1 + 3\left(\frac{1}{2}\right)}$$

$$= \frac{6 - 1}{2 + 3}$$

$$= 1$$

$$= \tan\frac{\pi}{4}$$

$$\therefore A - B = \frac{\pi}{4}$$

If tanB = 2 and tanC = 3 in ΔABC, then prove that tanA = 1.

➡ In ΔABC, A + B + C = π

$\therefore B + C = \pi - A$

$\therefore \tan(B + C) = \tan(\pi - A)$

$$\therefore \frac{\tan A + \tan C}{1 - \tan A \tan C} = -\tan A$$

$$\therefore \frac{2 + 3}{1 - (2)(3)} = -\tan A$$

$$\therefore \frac{5}{-5} = -\tan A$$

$\therefore -1 = -\tan A$

$\therefore \tan A = 1$

If $0 < A, B < \frac{\pi}{2}$, $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$, prove that $A + B = \frac{\pi}{4}$.

➡ $$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{a}{a+1} + \frac{1}{2a+1}}{1 - \frac{a}{a+1} \cdot \frac{1}{2a+1}}$$

$$= \frac{a(2a+1) + (a+1)}{(a+1)(2a+1) - a}$$

$$= \frac{2a^2 + a + a + 1}{2a^2 + 2a + a + 1 - a}$$

$$= \frac{2a^2 + 2a + 1}{2a^2 + 2a + 1} = 1$$

$$= \tan\frac{\pi}{4}$$

$$\therefore A + B = \frac{\pi}{4}$$

18. If $\alpha + \beta = \theta$, $\alpha - \beta = \phi$ and $\frac{\tan\alpha}{\tan\beta} = \frac{x}{y}$, then prove that $\frac{\sin\theta}{\sin\phi} = \frac{x + y}{x - y}$.

➡ $$\frac{\tan\alpha}{\tan\beta} = \frac{x}{y}$$

applying compouando and dividando.

$$\frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \frac{x + y}{x - y}$$

$$\therefore \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}} = \frac{x + y}{x - y}$$

$$\therefore \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{x + y}{x - y}$$

$$\therefore \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{x + y}{x - y}$$

$$\therefore \frac{\sin\theta}{\sin\phi} = \frac{x + y}{x - y}$$

19. If $\frac{\tan(A - B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$, then prove that $\tan A \tan B = \tan^2 C$ [April 2013]

➡ $$\frac{\tan(A - B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$$

$$\therefore 1 - \frac{\tan(A - B)}{\tan A} = \frac{\sin^2 C}{\sin^2 A}$$

$$\therefore 1 - \frac{\sin(A - B)\cos A}{\cos(A - B)\sin A} = \frac{\sin^2 C}{\sin^2 A}$$

$$\therefore \frac{\cos(A - B)\sin A - \sin(A - B)\cos A}{\cos(A - B)\sin A} = \frac{\sin^2 C}{\sin^2 A}$$

$$\therefore \frac{\sin[A - (A - B)]}{\cos(A - B)\sin A} = \frac{\sin^2 C}{\sin^2 A}$$

$$\therefore \frac{\sin B}{\cos(A - B)} = \frac{\sin^2 C}{\sin A}$$

$$\therefore \frac{\sin A \sin B}{\sin^2 C} = \cos(A - B)$$

$\therefore \frac{\sin A \sin B}{\sin^2 C} = \cos A\cos B - \sin A\sin B$

$\therefore \frac{1}{\sin^2 C} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \sin B}$

$\therefore \operatorname{cosec}^2 C = \cot A\cot B - 1$

$\therefore 1 + \operatorname{cosec}^2 C = \cot A\cot B$

$\therefore \cot^2 C = \cot A\cot B$

$\therefore \tan A\tan B = \tan^2 C$

20. If tan(A + B) = 3 and tan(A − B) = 2, then find tan2A and tan2B.

➠ $\tan[(A + B) + (A - B)]$

$= \frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B)\tan(A - B)}$

$\therefore \tan 2A = \frac{3 + 2}{1 - (3)(2)}$

$\therefore \tan 2A = \frac{5}{-5} = -1$

$\therefore \tan[(A + B) - (A - B)]$

$= \frac{\tan(A + B) - \tan(A - B)}{1 + \tan(A + B)\tan(A - B)}$

$\therefore \tan 2B = \frac{3 - 2}{1 + (3)(2)}$

$\therefore \tan 2B = \frac{1}{7}$

21. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\cdot\sin^2\alpha}$, then prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

➠ $\tan(\alpha - \beta) = (1 - n)\tan\alpha$

$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{\frac{\sin\alpha}{\cos\alpha} - \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}}{1 + \frac{\sin\alpha}{\cos\alpha}\cdot\frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}}$

$= \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha\cos^2\alpha}{\cos\alpha - n\sin^2\alpha\cos\alpha + n\sin^2\alpha\cos\alpha}$

$\therefore \tan(\alpha - \beta)$

$= \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha(1 - \sin^2\alpha)}{\cos\alpha}$

$= \frac{\sin\alpha - n\sin\alpha}{\cos\alpha}$

$= \frac{\sin\alpha}{\cos\alpha}(1 - n)$

$= \tan\alpha\,(1 - n)$

Thus, $\tan(\alpha - \beta) = (1 - n)\tan\alpha$

Do It yourself

❖ **(Based on Exercise 4.2, Example 11 to 17)**

(1) If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$ then prove that $A + B = \frac{\pi}{4}$.

(2) If $\tan A = x\tan B$ then prove that $\frac{\sin(A - B)}{\sin(A + B)} = \frac{x - 1}{x + 1}$.

(3) If $\cos A + \sin B = m$ and $\sin A + \cos B = n$ then prove that $2\sin(A + B) = m^2 + n^2 - 2$.

(4) If $\tan A + \tan B = a$ and $\cot A + \cot B = b$ then prove that $\cot(A - B) = \cot(A + B)$.

(5) If $\tan\alpha = x + 1$, $\tan B = x - 1$ then prove that $2\cot(\alpha - \beta) = x^2$.

(6) If $2\tan\beta + \cot\beta = \tan\alpha$ then prove that $\cot\beta = 2\tan(\alpha - \beta)$

(7) If $\sin B = 3\sin(2A + B)$ then prove that $2\tan A + \tan(A + B) = 0$.

(8) Prove : $\frac{\sin(x + \theta)}{\sin(x + \phi)} = \cos(\theta - \phi) + \cot(x + \phi)\cdot\sin(\theta - \phi)$

Exercise 4.3 [Textbook Page No. 81]

[illegible] as a sum or a difference :

[illegible] $\cos 3\theta$

[illegible] $+ \sin(7\theta - 3\theta)$

[illegible] $\sin 4\theta$

[illegible] $\frac{3\theta}{2}$

[illegible] $\sin\frac{\theta}{2}$

[illegible] $- \sin\left(\frac{5\theta}{2} - \frac{\theta}{2}\right)$

[illegible] $- \sin(2\theta)$

[illegible] $\sin 3\theta$

[illegible] $+ 3\theta) - \sin(5\theta - 3\theta)$

[illegible] $- \sin 2\theta$

[illegible] $\cos\frac{5\theta}{2}$

[illegible] $+ \sin\left(\frac{7\theta}{2} - \frac{5\theta}{2}\right)$

[illegible] $+ \sin(\theta)$

[illegible] $\cos 3\theta$

[illegible] $+ 3\theta^\circ) + \cos(11\theta - 3\theta)$

[illegible] $+ \cos 8\theta$

(6) $2\cos\frac{5\theta}{2}\cos\frac{3\theta}{2}$

➠ $= \cos\left(\frac{5\theta}{2} + \frac{3\theta}{2}\right) + \cos\left(\frac{5\theta}{2} - \frac{3\theta}{2}\right)$

$= \cos(4\theta) + \cos(\theta)$

(7) $\sin 9\theta \ \sin 11\theta$

➠ $= -\frac{1}{2}[-2\sin 11\theta \sin 9\theta]$

$= -\frac{1}{2}\left[\cos\left(\frac{20\theta}{2}\right) - \cos\left(\frac{2\theta}{2}\right)\right]$

$= -\frac{1}{2}[\cos 10\theta - \cos 2\theta]$

$= \frac{1}{2}[\cos 2\theta - \cos 10\theta]$

(8) $2\sin\frac{9\theta}{2}\sin\frac{7\theta}{2}$

➠ $= -\left[-2\sin\frac{9\theta}{2} - \sin\frac{7\theta}{2}\right]$

$= -\left[\cos\left(\frac{16\theta}{2}\right) - \cos\left(\frac{2\theta}{2}\right)\right]$

$= -[\cos 8\theta - \cos\theta]$

$= \cos\theta - \cos 8\theta$

(9) $2\sin\theta\cos\theta$

➠ $= 2\sin(\theta + \theta) + \cos(\theta - \theta)$

$= 2\sin 2\theta \cdot \cos\theta$

$= 2\sin 2\theta\ (1)$

$= 2\sin 2\theta$

Do It yourself

Express the following in the form of the sum or the product.

(Based on Exercise 4.3, Example 1)

1. $2\sin 5\theta \cos\theta$ **(Ans. $\sin 6\theta + \sin 4\theta$)**
2. $2\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$ **(Ans. $\sin 2\theta + \sin\theta$)**
3. $2\cos 7\theta \sin 5\theta$ **(Ans. $\sin 12\theta - \sin 2\theta$)**
4. $2\cos\frac{11\theta}{2}\sin\frac{3\theta}{2}$ **(Ans. $\sin 7\theta - \sin 4\theta$)**
5. $2\cos 7\theta \cos\theta$ **(Ans. $\cos 8\theta + \cos 6\theta$)**
6. $2\sin 7\theta \sin 3\theta$ **(Ans. $\cos 4\theta - \cos 10\theta$)**
7. $2\sin^2\theta$ **[Hint : $2\sin^2\theta = 2\sin\theta\sin\theta$]** **(Ans. $1 - \cos 2\theta$)**
8. $2\cos^2\theta$ **(Ans. $1 + \cos 2\theta$)**

2. Find the value :

(1) $2\sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{12}$

➡ $\cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)-\cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)$

$=\cos\left(\frac{4\pi}{12}\right)-\cos\left(\frac{6\pi}{12}\right)$

$=\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{2}\right)$

$=\frac{1}{2}-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-1}{2\sqrt{2}}$

(2) $2\sin\frac{5\pi}{12}\cdot\cos\frac{7\pi}{12}$

➡ $\sin\left(\frac{5\pi}{12}+\frac{7\pi}{12}\right)+\sin\left(\frac{5\pi}{12}-\frac{7\pi}{12}\right)$

$=\sin\left(\frac{12\pi}{12}\right)+\sin\left(\frac{-2\pi}{12}\right)$

$=\sin\pi-\sin\left(\frac{\pi}{6}\right)$ $(\because \sin(-\theta)=-\sin\theta)$

$=0+\frac{1}{2}=\frac{1}{2}$

(3) $2\cos\frac{\pi}{12}\cdot\sin\frac{5\pi}{12}$

➡ $2\sin\frac{5\pi}{12}\cos\frac{\pi}{12}$

$=\sin\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)+\sin\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)$

$=\sin\left(\frac{6\pi}{12}\right)+\sin\left(\frac{4\pi}{12}\right)$

$=\sin\frac{\pi}{2}+\sin\frac{\pi}{3}$

$=1+\frac{\sqrt{3}}{2}$

$=\frac{2+\sqrt{3}}{2}$

(4) $2\cos\frac{5\pi}{12}\cdot\cos\frac{7\pi}{12}$

➡ $\cos\left(\frac{5\pi}{12}+\frac{7\pi}{12}\right)+\cos\left(\frac{5\pi}{12}-\frac{7\pi}{12}\right)$

$=\cos\left(\frac{12\pi}{12}\right)+\cos\left(\frac{-2\pi}{12}\right)$

$=\cos\pi+\cos\frac{\pi}{6}$ $(\because \cos(-\theta)=\cos\theta)$

$=-1+\frac{\sqrt{3}}{2}$

$=\frac{\sqrt{3}-2}{2}$

(5) $8\cos15°\cdot\cos45°\cdot\cos75°$

➡ $\cos45°\cdot 4(2\cos75°\cos15°)$ (Rearrangement

$=\frac{1}{\sqrt{2}}\cdot 4\,[\cos(75+15)+\cos(75-$

$=\frac{4}{\sqrt{2}}[\cos90°+\cos60°]$

$=\frac{4}{\sqrt{2}}\left[0+\frac{1}{2}\right]$

$=\frac{4}{\sqrt{2}}\times\frac{1}{2}=\sqrt{2}$

(6) $8\sin10°\cdot\sin50°\cdot\sin70°$.

➡ $4(2\sin50\,\sin10)\sin70$ (Exchange of t

$=4[\cos(50-10)-\cos(50+10)]$ si

$=4[\cos40-\cos60]\sin70$

$=4\left[\cos40-\frac{1}{2}\right]\sin70.$

$=4\sin70\cos40-1\cdot\frac{1}{2}\sin70$

$=2[2\sin70\cos40]-2\sin70$

$=2[\sin(70+40)+\sin(70-40)]-$

$=2[\sin110°+\sin30°]-2\sin70$

$=2\left[\sin110°+\frac{1}{2}\right]-2\sin70$

$=2\sin110°+2\left(\frac{1}{2}\right)-2\sin70$

$=2\sin(180-70)°+1-2\sin70$

$=2\sin70+1-2\sin70$

$=1$